

Moderne Methoden der Datenverarbeitung in der Physik

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- Fehlerfortpflanzung I
- Fehlerfortpflanzung II, Zufallszahlen
- Maximum-Likelihood Methode, Gewichteter Mittelwert
- Methode der kleinsten Quadrate, Lineare Regression I
- Methode der kleinsten Quadrate, Lineare Regression II
- Fouriertransformation

Maximum-Likelihood-Methode \Leftrightarrow kleinste Quadrate

$$F(\mathbf{a}) = -\ln L(\mathbf{a}) = -\sum_{i=1}^n \ln f(x_i | \mathbf{a}) = \text{Minimum} \Leftrightarrow \frac{\partial F}{\partial a_k} = 0 \text{ und } \frac{\partial^2 F}{\partial a_k^2} > 0$$

Mittelwert einer Gauß-Verteilung

Die Gauß-Wahrscheinlichkeitsdichte ist $f(x_i | \mathbf{a}) = \frac{1}{\sqrt{2\pi}\sigma_i} \exp\left[-\frac{(x_i - \mathbf{a})^2}{2\sigma_i^2}\right]$

mit dem Mittelwert \mathbf{a} . Die negative Log-Likelihood-Funktion ist

$$F(\mathbf{a}) = \text{konst.}(\sigma_i) + \frac{1}{2} \sum_{i=1}^n \frac{(x_i - \mathbf{a})^2}{\sigma_i^2}$$

Das Minimum von $F(\mathbf{a})$ ist gleichbedeutend mit dem Minimum von:

$$\chi^2 = \sum_{i=1}^n \frac{(x_i - \mathbf{a})^2}{\sigma_i^2}$$

Entwickelt 1801 von C.F. Gauss für die Astronomie und Landvermessung

Methode der kleinste Quadrate

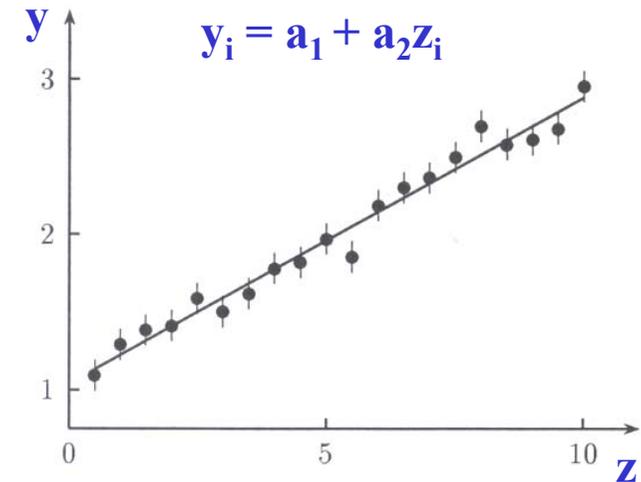
$$\chi^2 = \sum_{i=1}^n \frac{(x_i - a)^2}{\sigma_i^2}$$

An einer Reihe von Stützstellen z_i seien Werte für y_i vorgeben. Theoretisch vorhergesagt seien $y_t(z_i|a)$.

$$\chi^2 = \sum_{i=1}^n \frac{(y_i(z_i) - y_t(z_i | a))^2}{\sigma_i^2} = \text{Minimum} \Leftrightarrow \frac{d\chi^2}{da} = 0$$

Die Methode kann auf N Parameter a_k erweitert werden:

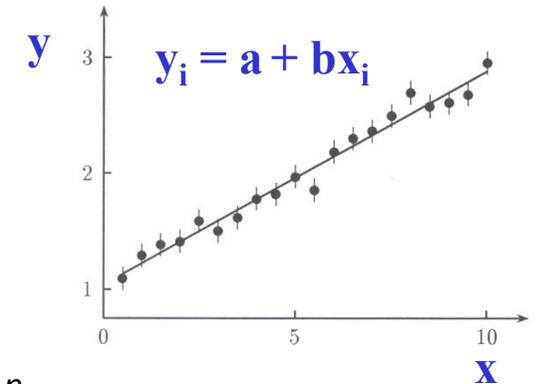
$$\frac{\partial \chi^2}{\partial a_k} = 0 \text{ für } k=1,2,\dots,N$$



Anpassung einer Geraden

Geradengleichung: $y(x) = y(x|a, b) = a + b \cdot x$

$$\chi^2(a, b) = \sum_{i=1}^n \frac{(y_i - a - bx_i)^2}{\sigma_i^2}$$



Das Minimum von $\chi^2(a, b)$ wird bestimmt durch:

$$\frac{\partial \chi^2}{\partial a} = 0 = \sum_{i=1}^n \frac{y_i - a - bx_i}{\sigma_i^2} \quad \Leftrightarrow \quad \sum_{i=1}^n \frac{y_i}{\sigma_i^2} = a \sum_{i=1}^n \frac{1}{\sigma_i^2} + b \sum_{i=1}^n \frac{x_i}{\sigma_i^2}$$

$$\frac{\partial \chi^2}{\partial b} = 0 = \sum_{i=1}^n \frac{x_i(y_i - a - bx_i)}{\sigma_i^2} \quad \Leftrightarrow \quad \sum_{i=1}^n \frac{x_i y_i}{\sigma_i^2} = a \sum_{i=1}^n \frac{x_i}{\sigma_i^2} + b \sum_{i=1}^n \frac{x_i x_i}{\sigma_i^2}$$

mit den Abkürzungen:

$$S_y \equiv \sum_{i=1}^n \frac{y_i}{\sigma_i^2} \quad S_x \equiv \sum_{i=1}^n \frac{x_i}{\sigma_i^2} \quad S_{xy} \equiv \sum_{i=1}^n \frac{x_i y_i}{\sigma_i^2} \quad S_{xx} \equiv \sum_{i=1}^n \frac{x_i x_i}{\sigma_i^2} \quad S \equiv \sum_{i=1}^n \frac{1}{\sigma_i^2}$$

vereinfacht sich das Gleichungssystem zu:

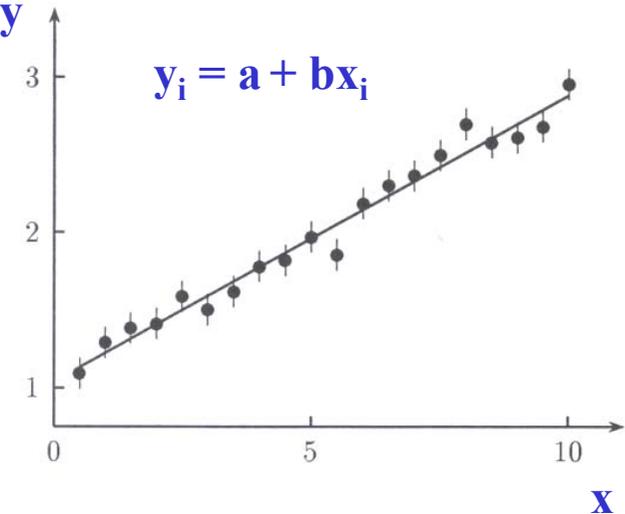
$$S_y = a \cdot S + b \cdot S_x \quad \text{und} \quad S_{xy} = a \cdot S_x + b \cdot S_{xx}$$

Anpassung einer Geraden

$$S_y = a \cdot S + b \cdot S_x \quad \text{und} \quad S_{xy} = a \cdot S_x + b \cdot S_{xx}$$

2. Gleichungen mit 2. Unbekannten: a, b

$$a = \frac{S_{xx} S_y - S_x S_{xy}}{S \cdot S_{xx} - S_x S_x} \quad \text{und} \quad b = \frac{S \cdot S_{xy} - S_x S_y}{S \cdot S_{xx} - S_x S_x}$$



$$S_y \equiv \sum_{i=1}^n \frac{y_i}{\sigma_i^2} \quad S_x \equiv \sum_{i=1}^n \frac{x_i}{\sigma_i^2} \quad S_{xy} \equiv \sum_{i=1}^n \frac{x_i y_i}{\sigma_i^2}$$

$$S_{xx} \equiv \sum_{i=1}^n \frac{x_i x_i}{\sigma_i^2} \quad S \equiv \sum_{i=1}^n \frac{1}{\sigma_i^2} \quad \Delta = S \cdot S_{xx} - S_x S_x$$

linreg.mw

*Praktikum.mws *fourier_0.mw *FourierTransformation_v01.mw *LineFit.mw *linfit.mw *weighted_mean.mw *maxLikelihood.mw *linreg.mw

Text Math Maple Input Monospaced 12

> # Stefan Schael, 09.12.2008
dv_maple/vorlesungen/08/linreg.mw
> restart;
 "Startup file \$HOME/.mapleinit executed." (1)

> fchiq:=(a,b)->sum((y[i]-(a+b*x[i]))^2/ey[i]^2,i=1..n);

$$fchiq := (a, b) \rightarrow \sum_{i=1}^n \frac{(y_i - a - b x_i)^2}{ey_i^2}$$
 (2)

> g11:=diff(fchiq(a,b),a);

$$g11 := \sum_{i=1}^n \left(-\frac{2(y_i - a - b x_i)}{ey_i^2} \right)$$
 (3)

> g12:=diff(fchiq(a,b),b);

$$g12 := \sum_{i=1}^n \left(-\frac{2(y_i - a - b x_i) x_i}{ey_i^2} \right)$$
 (4)

> res:=solve({g11=0,g12=0},{a,b});

$$res := \left\{ a = \frac{\left(\sum_{i=1}^n \frac{x_i y_i}{ey_i^2} \right) \left(\sum_{i=1}^n \frac{x_i}{ey_i^2} \right) - \left(\sum_{i=1}^n \frac{x_i^2}{ey_i^2} \right) \left(\sum_{i=1}^n \frac{y_i}{ey_i^2} \right)}{\left(\sum_{i=1}^n \frac{x_i}{ey_i^2} \right)^2 - \left(\sum_{i=1}^n \frac{1}{ey_i^2} \right) \left(\sum_{i=1}^n \frac{x_i^2}{ey_i^2} \right)}, b = -\frac{\left(\sum_{i=1}^n \frac{x_i}{ey_i^2} \right) \left(\sum_{i=1}^n \frac{y_i}{ey_i^2} \right) + \left(\sum_{i=1}^n \frac{1}{ey_i^2} \right) \left(\sum_{i=1}^n \frac{x_i y_i}{ey_i^2} \right)}{\left(\sum_{i=1}^n \frac{x_i}{ey_i^2} \right)^2 - \left(\sum_{i=1}^n \frac{1}{ey_i^2} \right) \left(\sum_{i=1}^n \frac{x_i^2}{ey_i^2} \right)} \right\}$$
 (5)

> subs(sum(1/ey[i]^2,i=1..n)=s, sum(x[i]/ey[i]^2,i=1..n)=sx, sum(y[i]/ey[i]^2,i=1..n)=sy, sum(x[i]*y[i]/ey[i]^2,i=1..n)=sxy, sum(x[i]^2/ey[i]^2,i=1..n)=sxx, res);

$$\left\{ a = \frac{sxy \, sx - sxx \, sy}{sx^2 - s \, sxx}, b = -\frac{-sx \, sy + s \, sxy}{sx^2 - s \, sxx} \right\}$$
 (6)

Ready

Server: 10 Memory: 1.12M Time: 0.06s Text Mode

Anpassung einer Geraden

Was sind die Fehler von a und b ?

$$a = a(y_i) \Rightarrow \sigma_a^2 = \sum_{i=1}^n \sigma_i^2 \left(\frac{\partial a}{\partial y_i} \right)^2 \quad \text{und entsprechend} \quad \sigma_b^2 = \sum_{i=1}^n \sigma_i^2 \left(\frac{\partial b}{\partial y_i} \right)^2$$

$$\frac{\partial a}{\partial y_i} = \frac{1}{\Delta} \left(S_{xx} \frac{\partial S_y}{\partial y_i} - S_x \frac{\partial S_{xy}}{\partial y_i} \right) = \frac{1}{\Delta} \left(S_{xx} \frac{1}{\sigma_i^2} - S_x \frac{x_i}{\sigma_i^2} \right) = \frac{1}{\Delta \sigma_i^2} (S_{xx} - S_x \cdot x_i)$$

$$\frac{\partial b}{\partial y_i} = \frac{1}{\Delta} \left(S \frac{\partial S_{xy}}{\partial y_i} - S_x \frac{\partial S_y}{\partial y_i} \right) = \frac{1}{\Delta} \left(S \frac{x_i}{\sigma_i^2} - S_x \frac{1}{\sigma_i^2} \right) = \frac{1}{\Delta \sigma_i^2} (S \cdot x_i - S_x)$$

$$\Rightarrow \sigma_a^2 = \frac{S_{xx}}{\Delta} = \frac{S_{xx}}{S \cdot S_{xx} - S_x S_x}$$

$$\sigma_b^2 = \frac{S}{\Delta} = \frac{S}{S \cdot S_{xx} - S_x S_x}$$

$$S_y \equiv \sum_{i=1}^n \frac{y_i}{\sigma_i^2} \quad S_x \equiv \sum_{i=1}^n \frac{x_i}{\sigma_i^2} \quad S_{xy} \equiv \sum_{i=1}^n \frac{x_i y_i}{\sigma_i^2}$$

$$S_{xx} = \sum_{i=1}^n \frac{x_i x_i}{\sigma_i^2} \quad S \equiv \sum_{i=1}^n \frac{1}{\sigma_i^2} \quad \Delta = S \cdot S_{xx} - S_x S_x$$

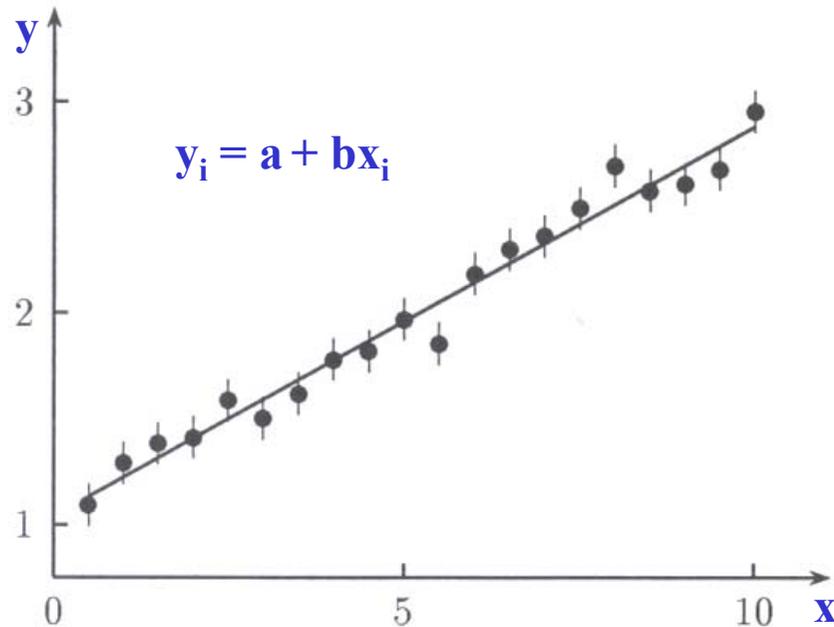
Anpassung einer Geraden

Was ist die Kovarianz zwischen a und b ?

$$\sigma_{ab} = \text{Cov}(a, b) = \sum_{i=1}^n \sigma_i^2 \frac{\partial a}{\partial y_i} \frac{\partial b}{\partial y_i} = -\frac{S_x}{\Delta} = -\frac{S_x}{S \cdot S_{xx} - S_x S_y}$$

und der Korrelationskoeffizient ist:

$$\rho(a, b) = \frac{\sigma_{ab}}{\sigma_a \sigma_b} = \frac{-S_x}{\sqrt{S \cdot S_{xx}}}$$



$$S_y \equiv \sum_{i=1}^n \frac{y_i}{\sigma_i^2} \quad S_x \equiv \sum_{i=1}^n \frac{x_i}{\sigma_i^2} \quad S_{xy} \equiv \sum_{i=1}^n \frac{x_i y_i}{\sigma_i^2}$$

$$S_{xx} = \sum_{i=1}^n \frac{x_i x_i}{\sigma_i^2} \quad S \equiv \sum_{i=1}^n \frac{1}{\sigma_i^2} \quad \Delta = S \cdot S_{xx} - S_x S_y$$

$$a = \frac{S_{xx} S_y - S_x S_{xy}}{\Delta} \quad \text{und} \quad b = \frac{S \cdot S_{xy} - S_x S_y}{\Delta}$$

$$\sigma_a^2 = \frac{S_{xx}}{\Delta} \quad \sigma_b^2 = \frac{S}{\Delta} \quad \rho(a, b) = \frac{-S_x}{\sqrt{S \cdot S_{xx}}}$$

LineFit.mw

Praktikum.mws fourier_0.mw "FourierTransformation_v01.mw" "LineFit.mw"

Text Math 2D Output Times New Roman 12 B I U

Variables

| Var... | Value |
|-----------|-----------|
| a | -2.29... |
| chiq | 11.97... |
| CovM... | 2 x 2... |
| f | t -> a... |
| fchiq | (xa, x... |
| LIB | "/Use... |
| libname | "/Libr... |
| Linear... | proc (... |
| m | -1.98... |
| p1A | PLOT(... |
| p1M | PLOT(... |
| p2A | PLOT(... |
| p2M | PLOT(... |
| p3A | CURV... |
| p3M | CURV... |
| p4A | CURV... |
| p4M | CURV... |
| p5A | CURV... |
| p5M | CURV... |
| Plot1 | PLOT(... |
| Plot2 | PLOT(... |
| PlotS1 | PLOT(... |
| PlotS2 | PLOT(... |
| Result | [-2.2... |
| rho | -.862... |

Handwriting
Expression
Units (SI)
Units (FPS)
Common Symbols
Matrix
Components
Greek
Arrows
Relational
Relational Round
Negated
Large Operators
Operators
Open Face
Fraktur
Script
Miscellaneous

```

> # Stefan Schael, 20.05.2012
  # dv_maple/vorlesung/08/linfit.mw
> restart:
                                     "Startup file $HOME/.mapleinit executed."
> with(plots): with(Statistics): with(plottools):
> xvec := [0,1,2,3,4,5,6]:
> yvec := [4.8, 3.1, 1.2, -1.2, -3.1, -4.8, -6.9]:
> sig := [0.13,0.08,0.1,0.25,0.09,0.1,0.06]:
> ErrorPlot(yvec, xcoords=xvec,yerrors=sig, title = "Error Plot");

```

```

> sx := sum(xvec[k]/sig[k]^2,k=1..nops(xvec)):
> sy := sum(yvec[k]/sig[k]^2,k=1..nops(xvec)):
> s := sum(1/sig[k]^2,k=1..nops(sig)):
> sxx := sum(xvec[k]^2/sig[k]^2,k=1..nops(xvec)):
> sxy := sum(xvec[k]*yvec[k]/sig[k]^2,k=1..nops(xvec)):
> a := (sxx*sy-sx*sxy)/(s*sxx-sx*sx);
> sa := evalf(sqrt(sxx/(s*sxx-sx*sx)));
                                     a := 5.013320255
                                     sa := 0.06840285787
> m := (s*sxy-sx*sy)/(s*sxx-sx*sx);
> sm := evalf(sqrt(s/(s*sxx-sx*sx)));
                                     m := -1.985339202
                                     sm := 0.01602266507

```

Ready

Server: 6 Memory: 1.12M Time: 0.07s Math Mode

LineFit.mw

$a := 5.013320255$
 $sa := 0.06840285787$ (2)

```

> m := (s*sxy-sx*sy)/(s*sxx-sx*sx);
sm := evalf(sqrt(s/(s*sxx-sx*sx)));

```

$m := -1.985339202$
 $sm := 0.01602266507$ (3)

```

> f := t->a+m*t;

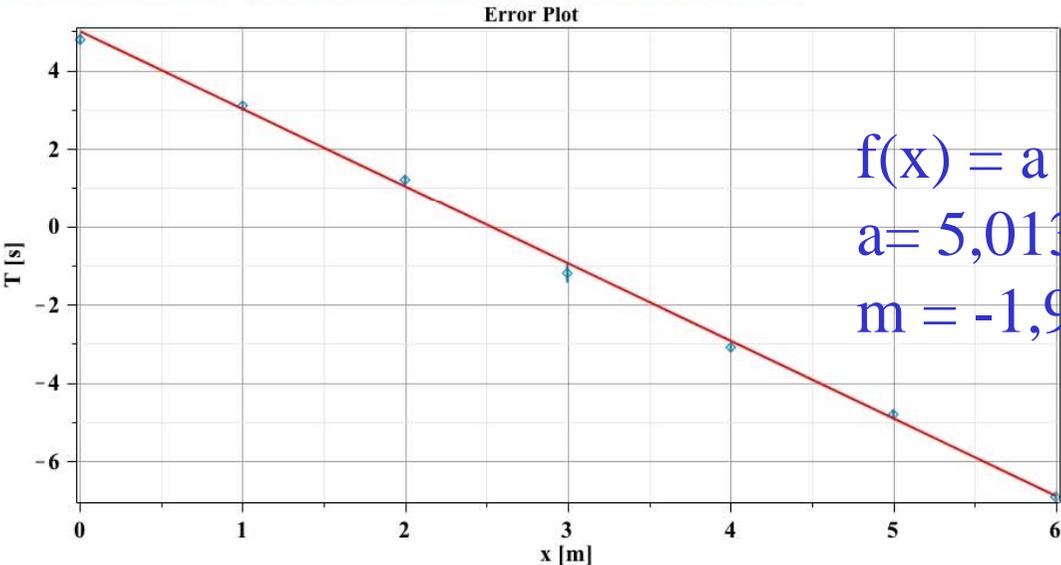
```

$f := t \rightarrow a + m t$ (4)

```

> Plot1 := ErrorPlot(yvec, xcoords=xvec,yerrors=sig, title = "Error Plot");
> Plot2 := plot(f(t),t=0..6);
> display(Plot1,Plot2,labels=["x [m]", "T [s]"],gridlines=true);

```



$f(x) = a + m \cdot x$
 $a = 5,013 \pm 0,068$
 $m = -1,985 \pm 0,016$

▼ Korrelation & Kovarianz-Matrix

```

> rho := -sx/sqrt(s*sxx);

```

$\rho := -0.8621622122$

```

> CovMatrix := Matrix([[sa^2, rho*sa*sm], [rho*sa*sm, sm^2]]);

```

$CovMatrix := \begin{bmatrix} 0.004678950965 & -0.0009449264061 \\ -0.0009449264061 & 0.0002567257959 \end{bmatrix}$

$\rho(a,m) = \frac{\sigma_{am}}{\sigma_a \sigma_m} = \frac{-S_x}{\sqrt{S \cdot S_{xx}}} \quad (1.1)$
 (1.2)

LineFit.mw

Praktikum.mws fourier_0.mw *FourierTransformation_v01.mw *LineFit.mw

Text Math P Heading 1 Times New Roman 18

Eigene Prozedur für die Lineare Regression

```

> # Eigene Prozedur für die Lineare Regression
# Geradengleichung: y = a + m*x
LineareRegression:=proc(xListe::list,yListe::list,eyListe::list)
local n, s, sx, sy, sxx, sxy, delta, i,
      a, m, ea, em, rho, chiq,
      p1, p2;
if (nops(xListe)<>nops(yListe)) then
printf("nops(xListe)=%d  nops(yListe)=%d \n",nops(xListe),nops(yListe));
error("xListe und yListe müssen die gleiche Anzahl von Elementen haben !");
end if;
n      := nops(xListe);
s      := sum(1.0/eyListe[i]^2,i=1..n);
sx     := sum(xListe[i]/eyListe[i]^2,i=1..n);
sy     := sum(yListe[i]/eyListe[i]^2,i=1..n);
sxx    := sum(xListe[i]^2/eyListe[i]^2,i=1..n);
sxy    := sum(xListe[i]*yListe[i]/eyListe[i]^2,i=1..n);
delta  := evalf(s*sxx-sx*sx);
a      := evalf((sxx*sy-sx*sxy)/delta);
m      := evalf((s*sxy-sx*sy)/delta);
ea     := evalf(sqrt(sxx/delta));
em     := evalf(sqrt(s/delta));
rho    := evalf(-sx/sqrt(s*sxx));
chiq   := sum((yListe[i]-(a+m*xListe[i]))^2/eyListe[i]^2,i=1..n);
printf("Geradengleichung: y = a + b*x \n");
printf("a          = %12.4e +/- %12.4e \n",eval(a),eval(ea));
printf("m          = %12.4e +/- %12.4e \n",eval(m),eval(em));
printf("rho         = %12.4e \n",eval(rho));
printf("chi^2/nDoF = %12.4e \n",eval(chiq)/(n-2.0));
return([a, ea, m, em, chiq, rho]);
end proc;

> Result := LineareRegression(xvec,yvec,sig):
Geradengleichung: y = a + b*x
a          = 5.0133e+00 +/- 6.8403e-02
m          = -1.9853e+00 +/- 1.6023e-02
rho        = -8.6216e-01
chi^2/nDoF = 2.3950e+00

```

Ready

Server: 6 Memory: 1.12M Time: 0.07s Text Mode

Nach der linearen Koordinatentransformation $x_i' = x_i - \frac{S_x}{S}$ gilt: $\rho = 0$

LineFit.mw

Praktikum.mws fourier_0.mw *FourierTransformation_v01.mw *LineFit.mw

Text Math Maple Input Monospaced 12

▼ Korrelation zwischen a und m durch Transformation minimieren

```

> xvecS := map(x->x-sx/s, xvec):
> Result := LinearRegression(xvecS, yvec, sig):
Geradengleichung: y = a + b*x
a      = -2.2941e+00 +/- 3.4655e-02
m      = -1.9853e+00 +/- 1.6023e-02
rho    = 0.0000e+01
chi^2/nDoF = 2.3950e+00
> a      := Result[1]:
SigmaA  := Result[2]:
m       := Result[3]:
SigmaM  := Result[4]:
> PlotS1 := ErrorPlot(yvec, xcoords=xvecS, yerrors=sig, title = "Error Plot"):
> PlotS2 := plot(a+m*t, t=-4..4, color=blue):
> display(PlotS1, PlotS2, labels=["x [m]", "T [s]"], gridlines=true);

```

$f(x) = a + m \cdot x$
 $a = 2,294 \pm 0,035$
 $m = -1,985 \pm 0,016$

Error Plot

T [s]

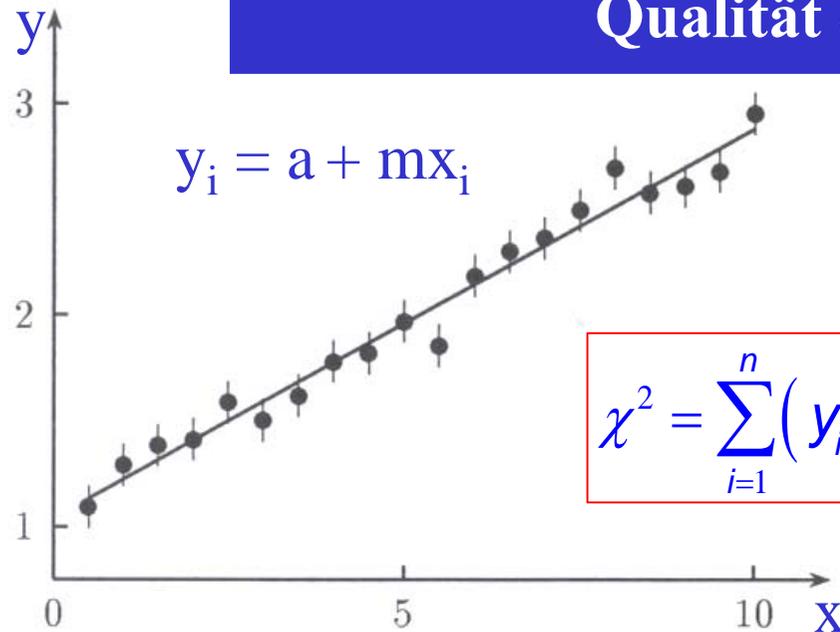
DV in der Physik, SS⁰2012, Vorlesung 8

12

Prof. Dr. Stefan Schael,
RWTH Aachen

Ready Server: 6 Memory: 1.12M Time: 0.07s Text Mode

Qualität der Anpassung



$$\chi^2 = \sum_{i=1}^n (y_i - f(x_i))^2 / \sigma_i^2 = \sum_{i=1}^n (y_i - (a + m \cdot x_i))^2 / \sigma_i^2$$

Wie beurteile ich, ob die Gerade meine Messwerte gut beschreibt ? $\leq \chi^2$

- Je mehr Messwerte n ich habe, um so größer wird die mittlere quadratische Abweichung zwischen den Messwerten und der Funktion sein.
- Je mehr freie Parameter ich habe, um so kleiner wird mein χ^2 werden.
- **Freiheitsgrad = Anzahl der Messwerte – Anzahl der freien Parameter**
Ein χ^2 von $1/\text{Freiheitsgrad}$ heißt, dass die Funktion die Messwerte im Rahmen der Fehler gut beschreibt.
Beispiel: 4 Messwerte, Funktion=Gerade=>2 freie Parameter => 2 Freiheitsgrade

MAPLE Bibliotheken

Prozeduren, die wir immer wieder verwenden wollen, können wir in eigenen MAPLE Bibliotheken ablegen:

```
> restart:
> dv_maple:=table():
> # Weighted mean
> dv_maple[weigthed_mean]:=proc(data::list,sig::list)
  local nk, k, mean, e_mean, sig2;
  nk      := nops(data):
  sig2    := add(1/sig[k]^2,k=1..nk);
  mean    := add(data[k]/sig[k]^2,k=1..nk)/sig2;
  e_mean  := 1/sqrt(sig2);
  return [mean,e_mean];
end proc:
> save dv_maple, "/Users/stefan/lib/dv_maple.m";
```

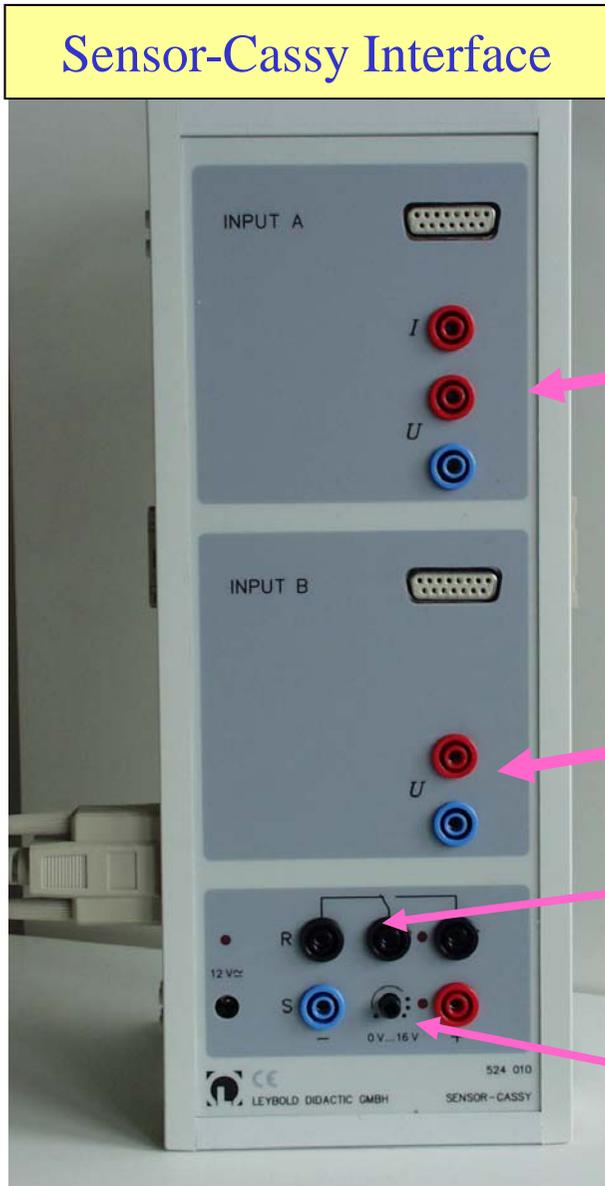
$$a = \frac{\sum_{i=1}^n x_i \cdot 1 / \sigma_i^2}{\sum_{i=1}^n 1 / \sigma_i^2}$$
$$\sigma_a = \frac{1}{\sqrt{\sum_{i=1}^n 1 / \sigma_i^2}}$$

und anschließend, genau wie die MAPLE internen Pakete verwenden:

```
> restart:
> libname:=libname, "/Users/stefan/lib/":
> with(dv_maple);
```

Messgerät im Physikalischen Grundpraktikum

Sensor-Cassy Interface



Eingang A (I,U)

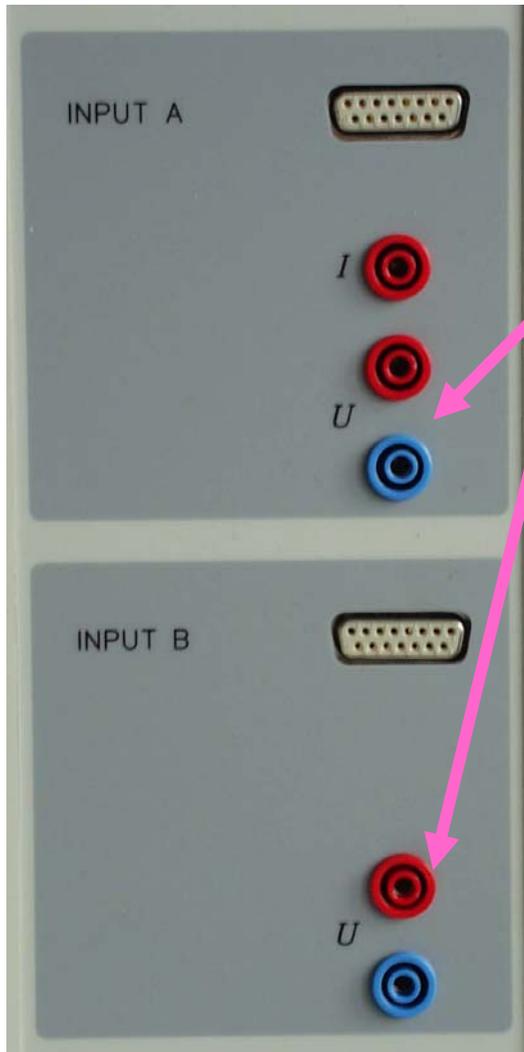
Eingang B (U)

Relais R

Spannungsquelle S (0 – 16V)

Messgerät im Physikalischen Grundpraktikum

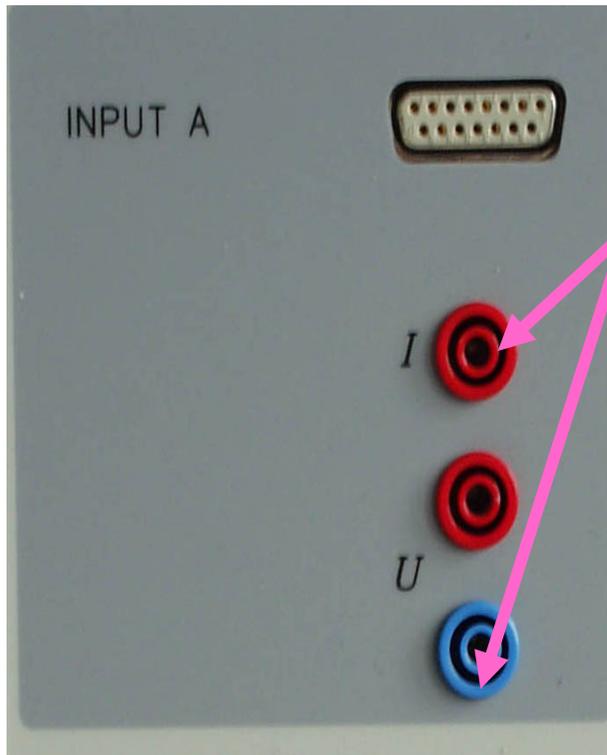
5 analoge Eingänge



2 analoge Spannungseingänge A und B:

- Auflösung: 12 Bit ($2^{12} = 4096$)
- Messbereiche: $\pm 0,3/1/3/10/30/100$ V
- Digitalisierung: $\pm 0,15$ mV/.../ 48,8mV
- Eingangswiderstand: 1 M Ω
- Abtastrate: max. 200.000 Werte/s
(=100.000 Werte/s pro Eingang)
- Anzahl Messwerte: max. 32000
(= 16000/ Eingang)

Eingang A:



1 analoger Stromeingang :

- Messbereiche: $\pm 0,1/0,3/1/3$ A
- Digitalisierung: $\pm 0,05$ mA/ ... / 1,5 mA
- Eingangswiderstand: $< 0,5 \Omega$

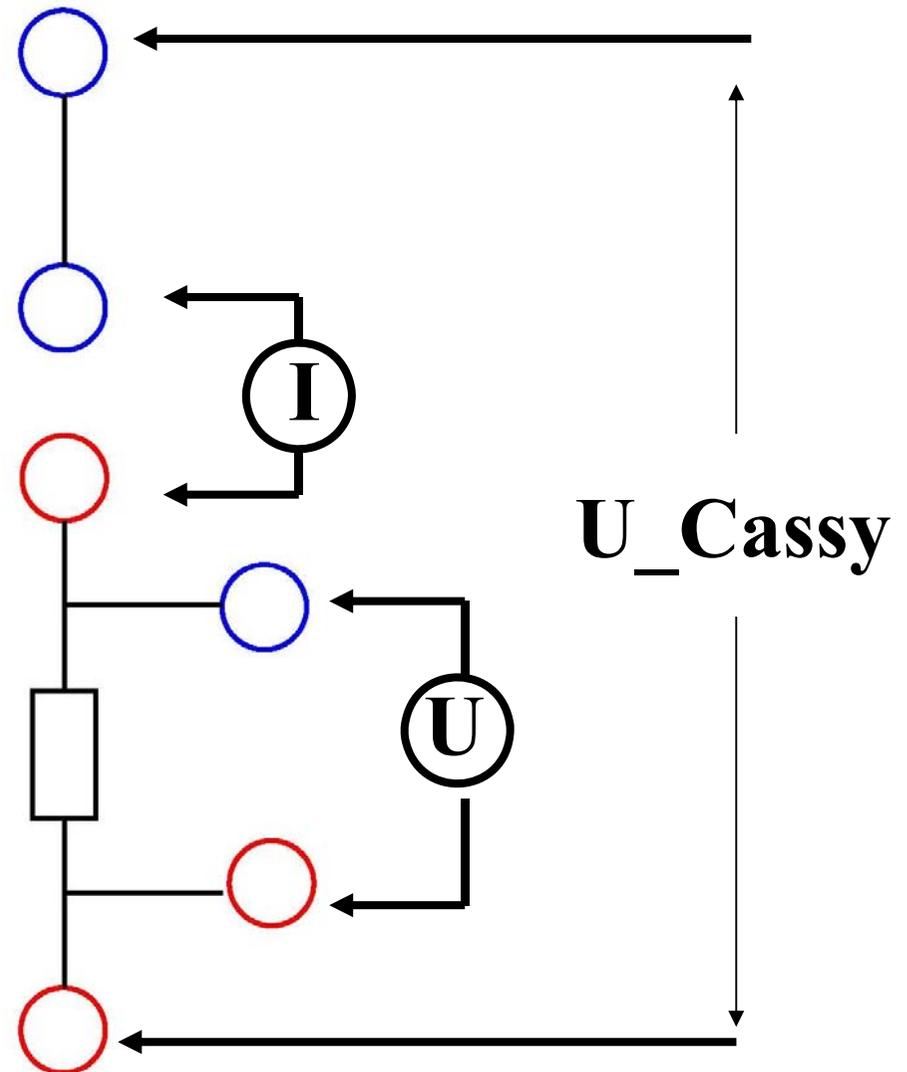
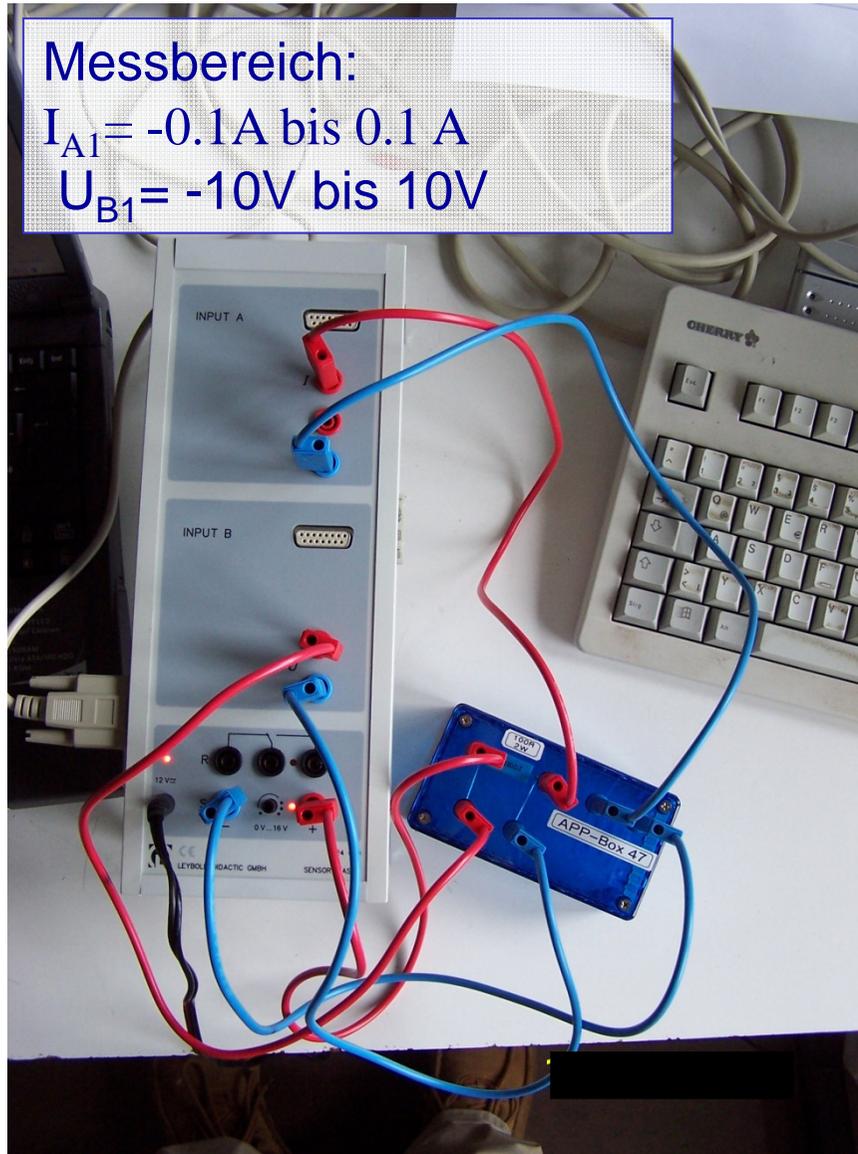
Strom- Spannungsmessung im PGP



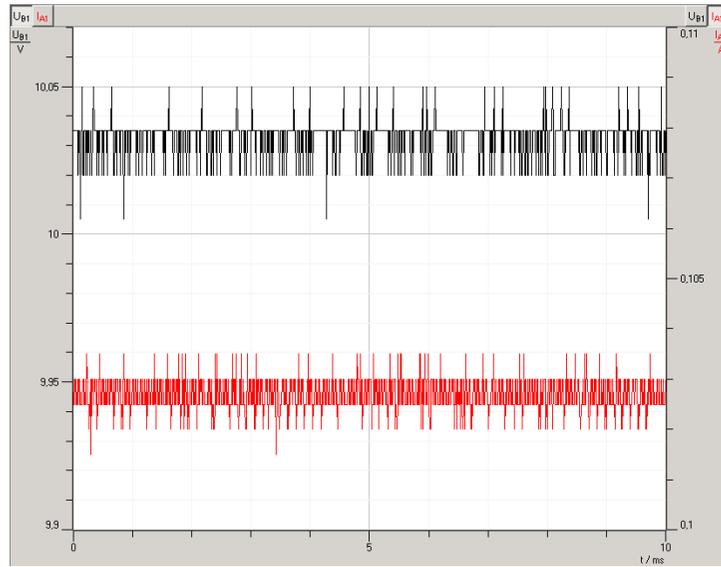
Messbereich:

$$I_{A1} = -0.1 \text{ A bis } 0.1 \text{ A}$$

$$U_{B1} = -10 \text{ V bis } 10 \text{ V}$$



Cassy-Lab Datei



```

100R_U_I_t_c.lab - XEmacs
File Edit View Cmds Tools Options Buffers
Open Dired Save Print Cut Copy Paste Undo Spell Replace Mail Info Compile Debug News
100R_U_I_t_c.lab
CL4
180 0.1
Index
n
0 16001 5000 0 0 0 0 1 0 0 0 0 0 16001 1 1
Zeit
t
ms
0 0.16 0.05 0 5 0 1 1 0 0 0 0 0 0 1 1
Frequenz
f
Hz
0 50000 10000 0 0 0 0 1 16000 0 0 0 0 0 1 1
2 1
0 1 0 1
1 0 0 0 1 1 3 0 0 0 0 0 0 0 0 2500 0 0 0 1 307 98 297 140 0.5 0 0 1
Strom
I_A1
A
-0.1 0.1 0.05 0 4 0 0 0 0 1 1 0 0 0.0498 1 1
0 1 0 1
1 1 0 0 1 0 2 0 0 0 0 0 0 0 2500 0 0 0 1 608 102 297 140 0.5 0 0 1
Spannung
U_B1
V
-10 10 5 0 2 0 0 0 0 1 1 0 0 4.995 1 1
4 16001
1 0 0.04985 5
2 1E-5 0.0498 5.005
3 2E-5 0.04985 5
4 3E-5 0.04985 5
5 4E-5 0.04985 5
6 5E-5 0.04985 5
7 6E-5 0.04985 5
8 7E-5 0.04985 5
9 8E-5 0.04985 5
10 9E-5 0.04985 5

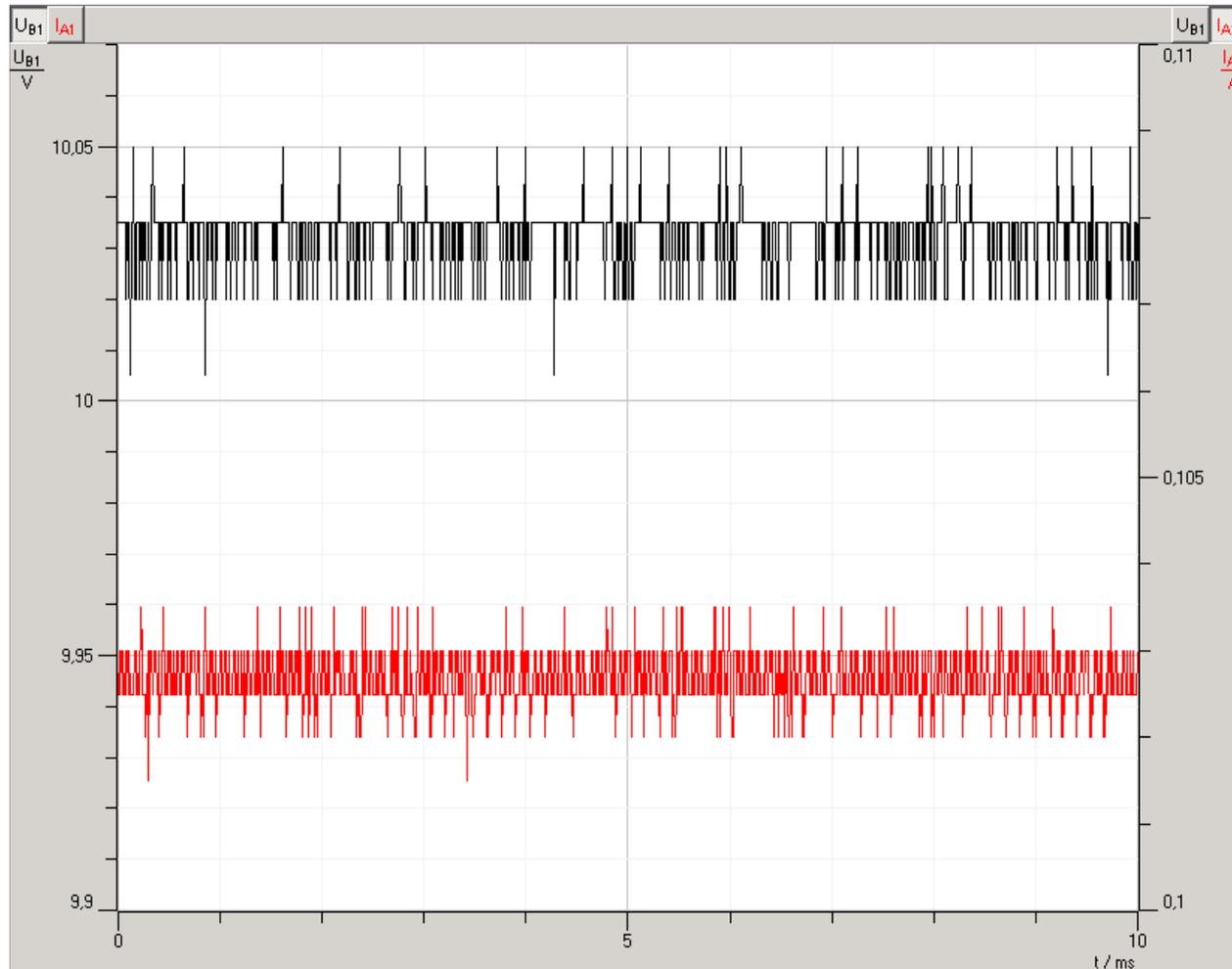
```



Prof. Dr. Stefan Schael,
RWTH Aachen

Cassy-Lab Datei ↔ Maple-Auswertung

Bestimmung des Ohmschen Widerstandes



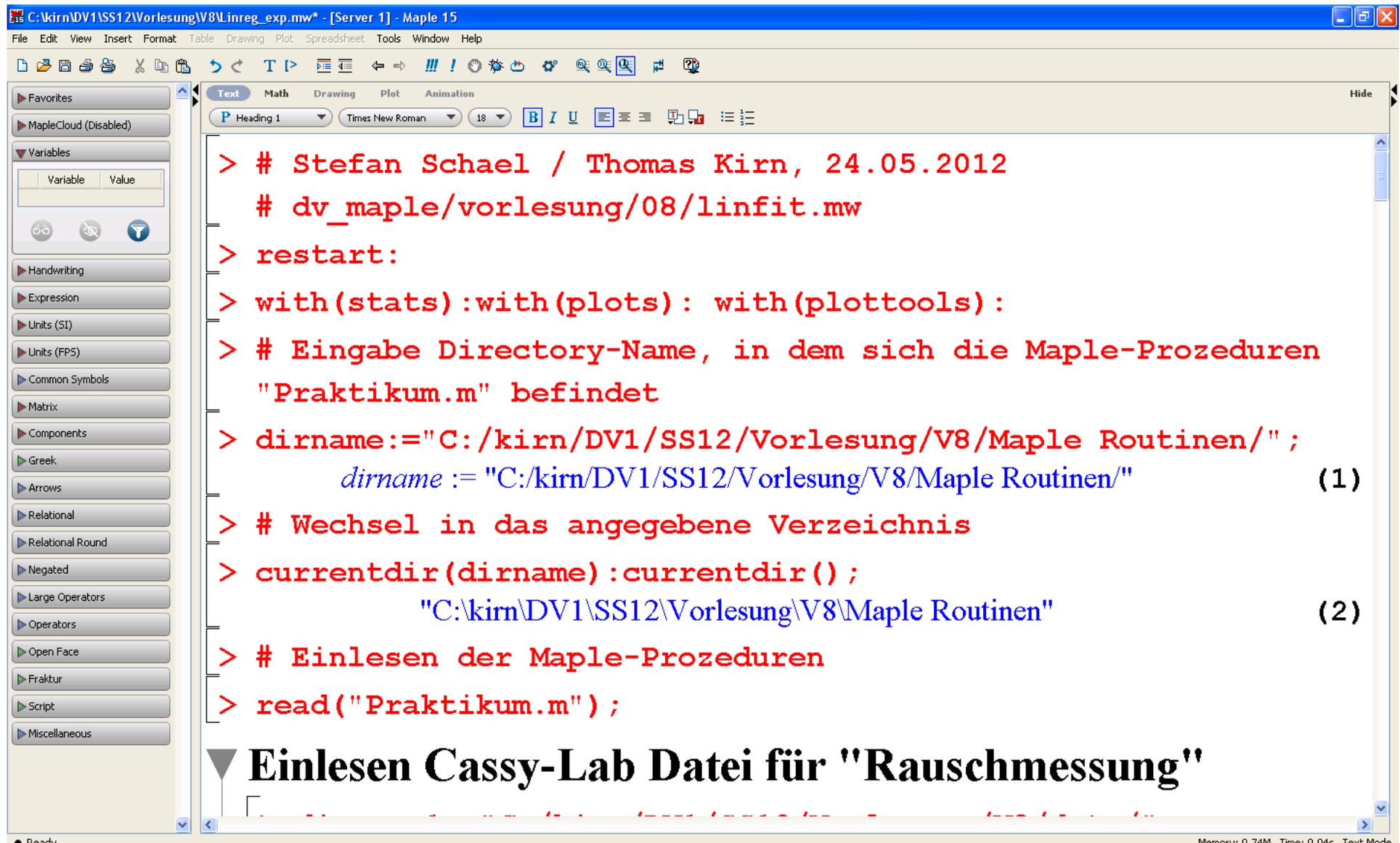
$$\langle x \rangle = \frac{1}{n} \cdot \sum_{i=1}^n x_i$$

$$\sigma = \sqrt{\frac{\sum_{i=1}^n (x_i - \langle x \rangle)^2}{n-1}}$$

$$\sigma_{\langle x \rangle} = \frac{\sigma}{\sqrt{n}}$$



Cassy-Lab Datei ↔ Maple-Auswertung



```
> # Stefan Schael / Thomas Kirn, 24.05.2012
# dv_maple/vorlesung/08/linfit.mw

> restart:

> with(stats):with(plots): with(plottools):

> # Eingabe Directory-Name, in dem sich die Maple-Prozeduren
  "Praktikum.m" befindet

> dirname:="C:/kirn/DV1/SS12/Vorlesung/V8/Maple Routinen/";
  dirname := "C:/kirn/DV1/SS12/Vorlesung/V8/Maple Routinen/" (1)

> # Wechsel in das angegebene Verzeichnis

> currentdir(dirname):currentdir();
  "C:\kirn\DV1\SS12\Vorlesung\V8\Maple Routinen" (2)

> # Einlesen der Maple-Prozeduren

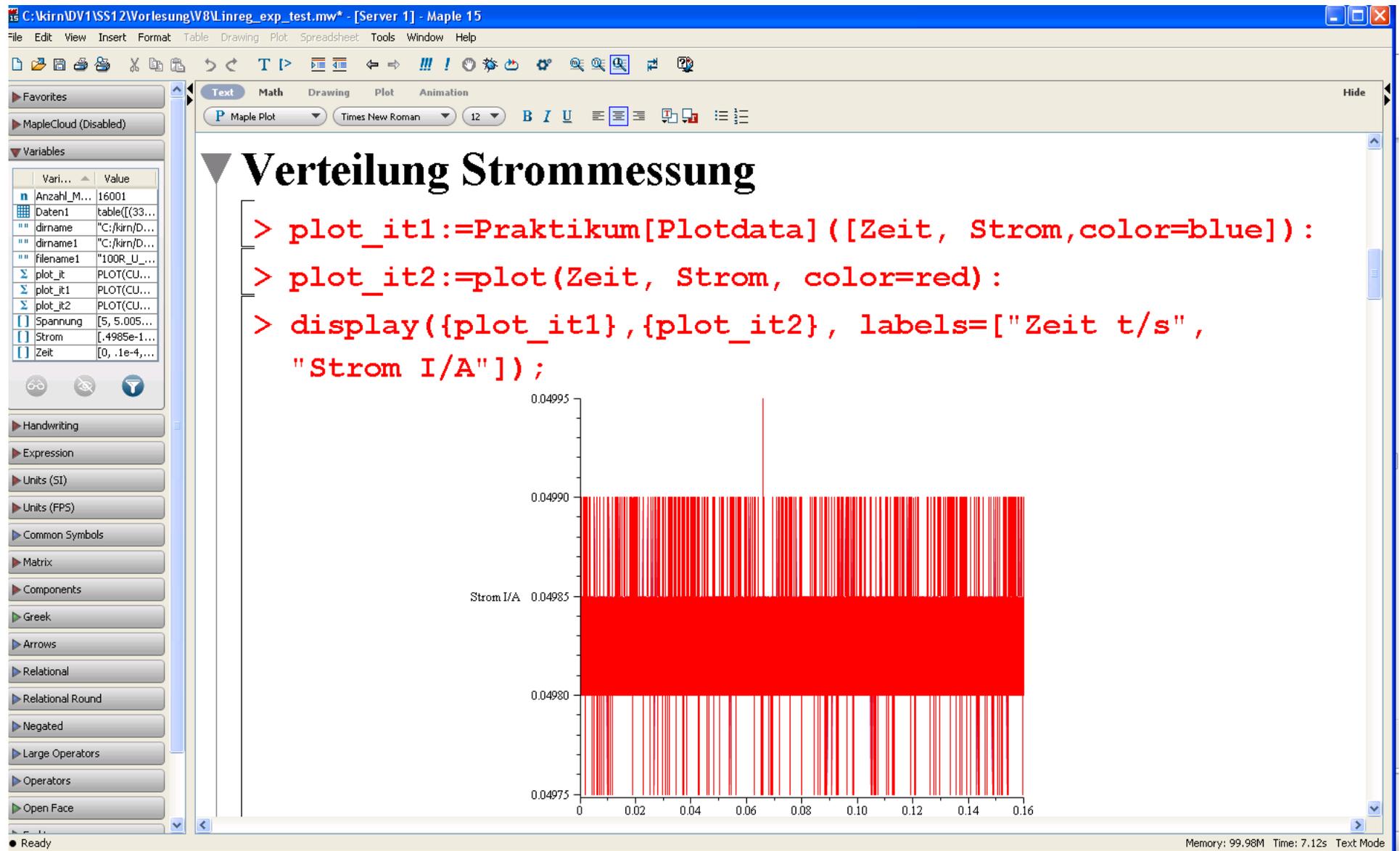
> read("Praktikum.m");
```

▼ **Einlesen Cassy-Lab Datei für "Rauschmessung"**

Einlesen Cassy-Lab Dateien in Maple mit Praktikumsbibliothek

```
C:\kirn\DV1\SS12\Vorlesung\V8\Linreg_exp.mw* - [Server 1] - Maple 15
File Edit View Insert Format Table Drawing Plot Spreadsheet Tools Window Help
Favorites
MapleCloud (Disabled)
Variables
Variable Value
Handwriting
Expression
Units (SI)
Units (FPS)
Common Symbols
Matrix
Components
Greek
Arrows
Relational
Relational Round
Negated
Large Operators
Operators
Open Face
Fraktur
Script
Miscellaneous
Text Math Drawing Plot Animation
Heading 1 Times New Roman 18 B I U
Einlesen Cassy-Lab Datei für "Rauschmessung"
> dirname1 := "C:/kirn/DV1/SS12/Vorlesung/V8/data/":
> currentdir(dirname1); currentdir();
"C:\kirn\DV1\SS12\Vorlesung\V8\Maple Routinen"
"C:\kirn\DV1\SS12\Vorlesung\V8\data" (1.1)
> # Geben Sie den File-Namen der Messdatei an
> filename1 := "100R_U_I_t_c.lab":
> # Einlesen des angegebenen Datenfiles und extrahieren
der einzelnen Messgroessen
> Daten1 := Praktikum[Datenlesen](filename1, xx, yy):
> Zeit := [seq(Daten1[i, 2], i=1..yy)]:
> Strom := [seq(Daten1[i, 3], i=1..yy)]:
> Spannung := [seq(Daten1[i, 4], i=1..yy)]:
> Anzahl_Messpunkte := nops(Spannung);
Anzahl Messpunkte := 16001 (1.2)
```

Verteilung der Strommesswerte



Verteilung der Strommesswerte

C:\Kirm\DV1\SS12\Worlesung\WB\Linreg_exp_test.mw* - [Server 1] - Maple 15

File Edit View Insert Format Table Drawing Plot Spreadsheet Tools Window Help

Text Math Drawing Plot Animation

Maple Plot Times New Roman 12 B I U

```

> MeanStrom:=Praktikum[Mean] (Strom) ;
Mean = 4.9834E-02 +/- 2.0754E-07 (Mittlerer quadratischer Fehler des Mittelwertes)
Rms = 2.6252E-05 (Standardabweichung)
MeanStrom := [0.04983428223, 2.075366333 10-7, 0.00002625235871] (2.1)

> with(Statistics):
> Mean (Strom) ;sqrt (Variance (Strom)) ;sqrt (Variance (Strom) /nops (Strom)) ;
0.04983428223
0.0000262523587495735
2.07536633582789 10-7 (2.2)

> plot_i:=Histogram(Strom, bincount=5):
> display({plot_i},labels=["I/A", "Haeufigkeit"],gridlines=true);

```

Memory: 99.98M Time: 7.12s Text Mode

Verteilung der Strommesswerte

C:\WirknDV1\SS12\Vorlesung\V8\Linreg_exp_test.mw* - [Server 1] - Maple 15

File Edit View Insert Format Table Drawing Plot Spreadsheet Tools Window Help

Text Math Drawing Plot Animation

Maple Input Monospaced 12 B I U

| V... | Value |
|-----------|-------------|
| Anzahl... | 16001 |
| Daten1 | table[(...] |
| dirname | "C:/kirm... |
| dirname1 | "C:/kirm... |
| fg | (x, xm, ... |
| fhi | PLOT(C... |
| filename1 | "100R... |
| i0 | .49834... |
| i1 | .49703... |
| i2 | .49965... |
| MaxStrom | .4995e-1 |
| MeanSt... | .49834... |
| MinStrom | .4975e-1 |
| plot_j | PLOT(P... |
| plot_it | PLOT(C... |
| plot_it1 | PLOT(C... |
| plot_it2 | PLOT(C... |
| si | .26252... |
| Spannung | [5, 5.00... |
| Strom | [.4985e... |
| Zeit | [0, .1e... |

```

> i0:=MeanStrom[1];si:=MeanStrom[3];
      i0 := 0.04983428223
      si := 0.00002625235871
(2.3)

> MinStrom:=min(seq(Strom[i],i=1..nops(Strom))); MaxStrom:=
max(seq(Strom[i],i=1..nops(Strom)));
      MinStrom := 0.04975
      MaxStrom := 0.04995
(2.4)

> i1:=i0-5*si;i2:=i0+5*si;
      i1 := 0.04970302044
      i2 := 0.04996554402
(2.5)

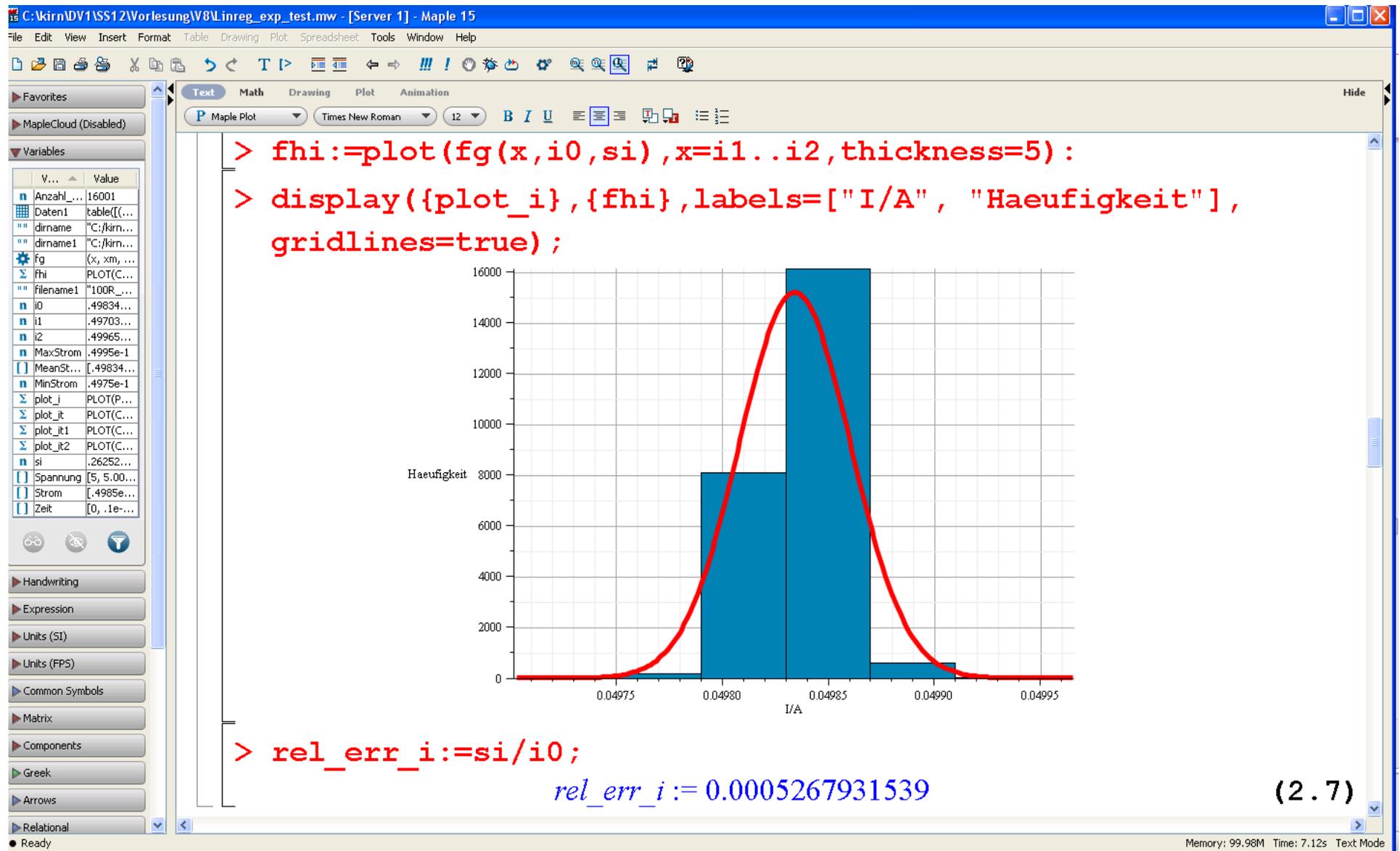
> fg:=(x,xm,sx)->1/sqrt(2*Pi*sx^2)*exp(-0.5*(x-xm)^2/sx^2);
      fg := (x, xm, sx) -> \frac{e^{-\frac{0.5(x-xm)^2}{sx^2}}}{\sqrt{2\pi sx^2}}
(2.6)

> fhi:=plot(fg(x,i0,si),x=i1..i2,thickness=5):

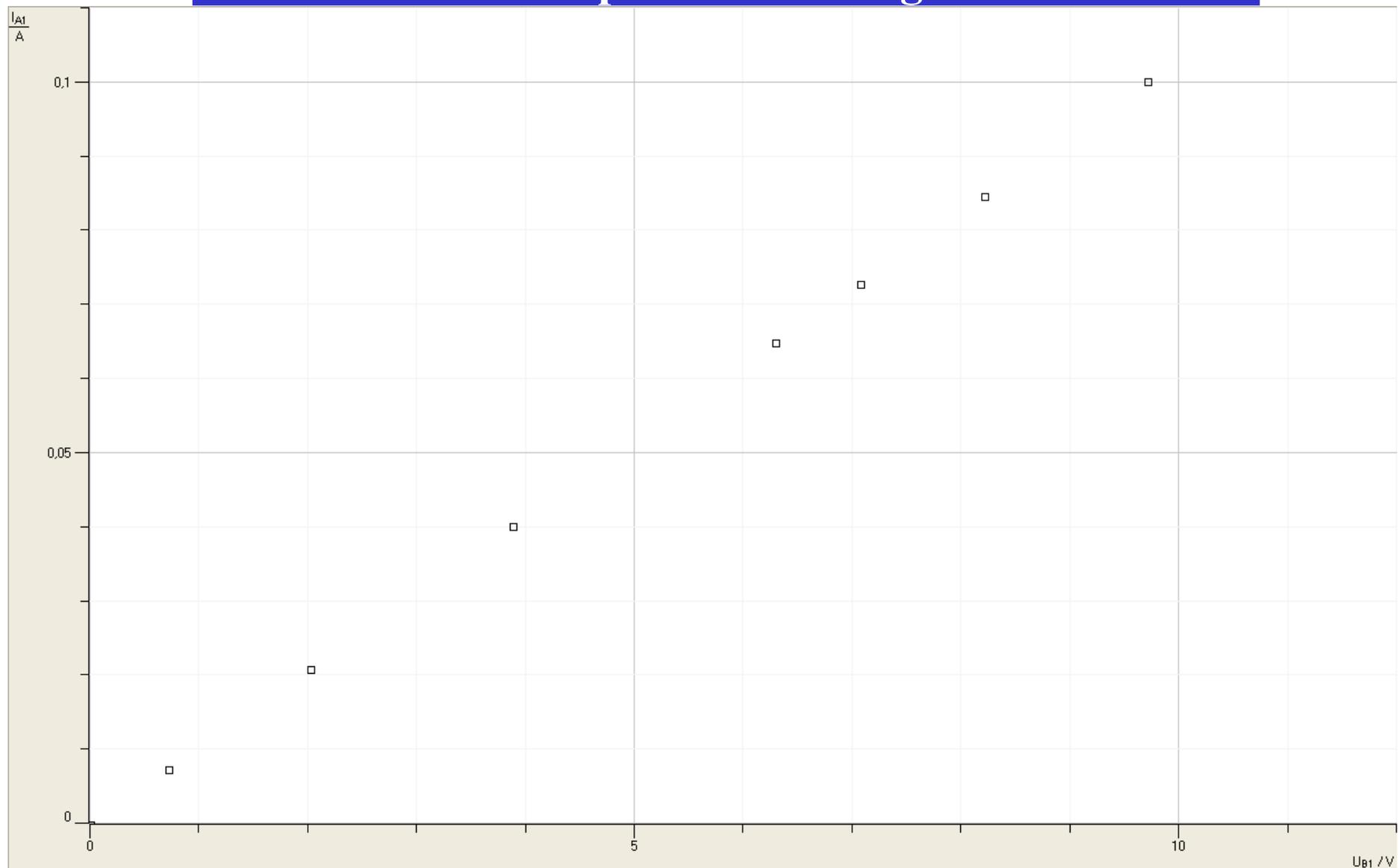
```

Memory: 99.98M Time: 7.12s Text Mode

Verteilung der Strommesswerte



Lineare Regression mit Cassy-Lab-Dateien via Maple-Auswertung



Lineare Regression mit Cassy-Lab-Dateien mit Maple

The screenshot shows the Maple 15 software interface. The main window displays a script titled "Einlesen Cassy-Lab Datei für lineare Regression". The script contains the following commands:

```
> dirname2 := "C:/kirn/DV1/SS12/Vorlesung/V8/data/":  
> currentdir(dirname2); currentdir();  
      "C:\kirn\DV1\SS12\Vorlesung\V8\data"  
      "C:\kirn\DV1\SS12\Vorlesung\V8\data" (3.1)  
> filename2 := "100R_U_I_m.lab":  
> Daten2 := Praktikum[Datenlesen](filename2, xx, yy):  
> Strom := [seq(Daten2[i, 3], i=1..yy)]:  
> Spannung := [seq(Daten2[i, 4], i=1..yy)]:  
> Anzahl_Messpunkte := nops(Spannung);  
      Anzahl_Messpunkte := 8 (3.2)  
>  
> xvec := Spannung:  
> yvec := Strom:  
> sig := rel_err_i * Strom:
```

The left sidebar shows a list of variables and their values, including "Anzahl_Messpunkte" (16001), "Daten1", "dirname", "dirname1", "fg", "phi", "filename1", "i0", "i1", "i2", "MaxStrom", "MeanStrom", "MinStrom", "plot_j", "plot_it", "plot_it1", "plot_it2", "si", "Spannung", "Strom", and "Zeit".

Lineare Regression mit Cassy-Lab-Dateien mit Maple

C:\kirmDV1\SS12\Vorlesung\W8\Linreg_exp_test.mw - [Server 1] - Maple 15

File Edit View Insert Format Table Drawing Plot Spreadsheet Tools Window Help

Text Math Drawing Plot Animation

Maple Plot Times New Roman 12 B I U

```
> Plot1:=ErrorPlot(yvec, xcoords=xvec,yerrors=sig, title =  
"Ohm'sches Gesetz",labels=["U/V", "I/A"],gridlines=true):  
> display(Plot1);
```

Ohm'sches Gesetz

| U/V | I/A |
|-----|-------|
| 0.5 | 0.008 |
| 2.0 | 0.021 |
| 3.8 | 0.040 |
| 6.3 | 0.065 |
| 8.2 | 0.085 |

Memory: 99.98M Time: 7.12s Text Mode

Lineare Regression mit Cassy-Lab-Dateien mit Maple

```

C:\Kirn\DV1\SS12\Worlesung\W8\Linreg_exp.mw* - [Server 1] - Maple 15
File Edit View Insert Format Table Drawing Plot Spreadsheet Tools Window Help
Favorites
MapleCloud (Disabled)
Variables
Variable Value
Handwriting
Expression
Units (SI)
Units (FPS)
Common Symbols
Matrix
Components
Greek
Arrows
Relational
Relational Round
Negated
Large Operators
Operators
Open Face
Fraktur
Script
Miscellaneous
Text Math Drawing Plot Animation
> sx := sum(xvec[k]/sig[k]^2,k=1..nops(xvec)):
  sy := sum(yvec[k]/sig[k]^2,k=1..nops(xvec)):
  s := sum(1/sig[k]^2,k=1..nops(sig)):
  sxx := sum(xvec[k]^2/sig[k]^2,k=1..nops(xvec)):
  sxy := sum(xvec[k]*yvec[k]/sig[k]^2,k=1..nops(xvec)):
> a := (sxx*sy-sx*sxy)/(s*sxx-sx*sx);
  sa := evalf(sqrt(sxx/(s*sxx-sx*sx)));
      a := -0.0004051249510
      sa := 1.359072355 10-7 (3)
> m := (s*sxy-sx*sy)/(s*sxx-sx*sx);
  sm := evalf(sqrt(s/(s*sxx-sx*sx)));
      m := 0.01034571791
      sm := 0.000002041201657 (4)
> f := t->a+m*t;
      f := t → a + m t (5)
> Plot2 := plot(f(t),t=0..10,thickness=2):
  
```

• Ready Memory: 0.74M Time: 0.10s Text Mode

Lineare Regression mit Cassy-Lab-Dateien mit Maple

C:\Kirm\DV1\SS12\Vorlesung\W8\Linreg_exp_test.mw - [Server 1] - Maple 15

File Edit View Insert Format Table Drawing Plot Spreadsheet Tools Window Help

Text Math Drawing Plot Animation

P Maple Plot Times New Roman 12 B I U

```
> f := t->a+m*t;
```

$$f := t \rightarrow a + mt \quad (5)$$

```
> Plot2 := plot(f(t), t=0..10, thickness=2);
```

```
> display(Plot1, Plot2, title = "Ohm'sches Gesetz", labels=
```

```
["U/V", "I/A"], gridlines=true);
```

| U/V (x) | I/A (y) |
|---------|---------|
| 0 | 0 |
| 1 | 0.01 |
| 2 | 0.02 |
| 4 | 0.04 |
| 6.5 | 0.065 |
| 7.5 | 0.075 |
| 8.5 | 0.085 |
| 10 | 0.10 |

Ohm'sches Gesetz

I/A

U/V

Ready Memory: 99.98M Time: 7.12s Text Mode

Lineare Regression mit Cassy-Lab-Dateien mit Maple

Korrelation & Kovarianz-Matrix

```
> rho := -sx/sqrt(s*sxx);
```

$$\rho := -0.2496781384 \quad (4.1)$$

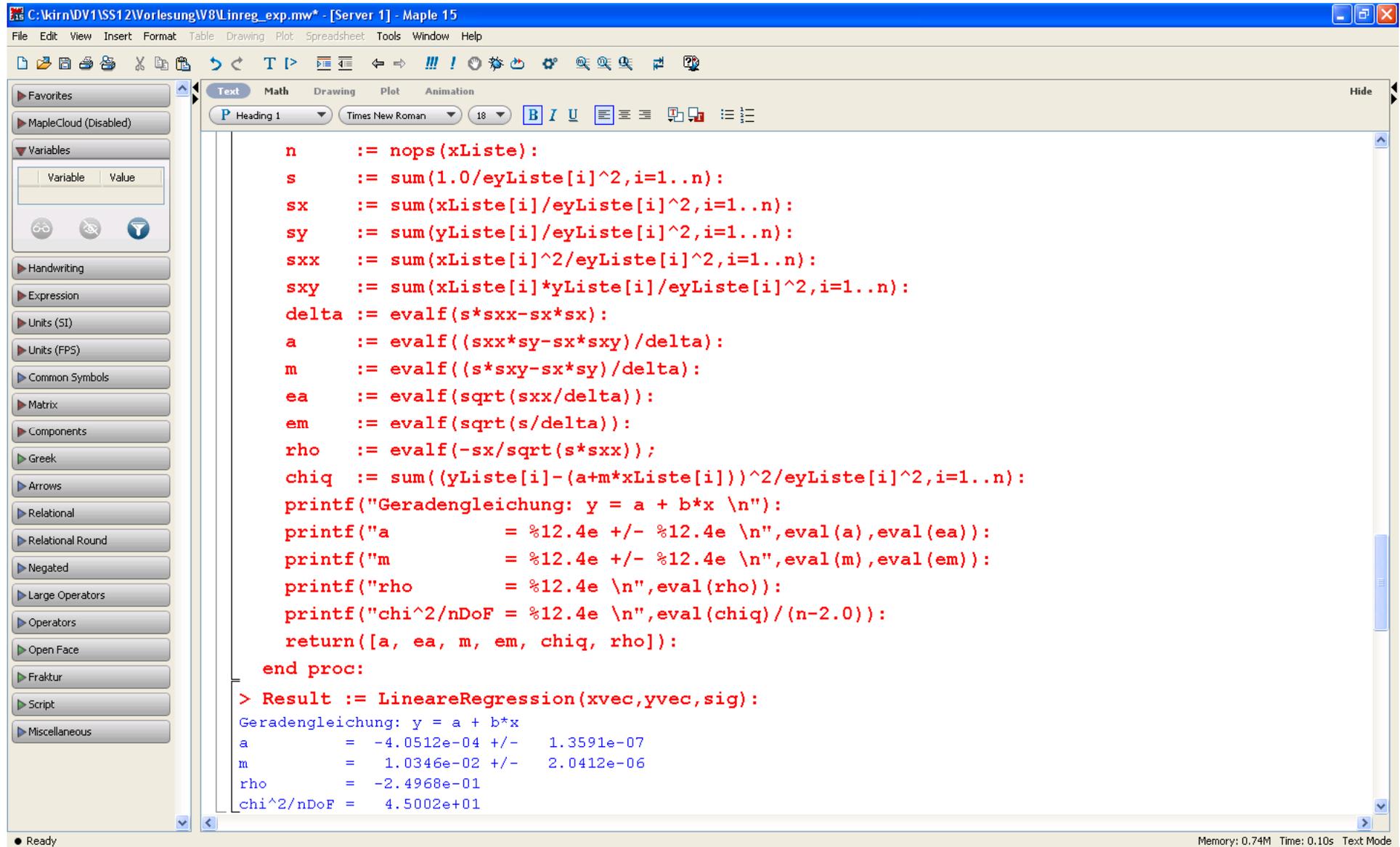
```
> CovMatrix := Matrix([[sa^2, rho*sa*sm], [rho*sa*sm, sm^2]]);
```

$$\text{CovMatrix} := \begin{bmatrix} 1.847077666 \cdot 10^{-14} & -6.926422963 \cdot 10^{-14} \\ -6.926422963 \cdot 10^{-14} & 4.166504205 \cdot 10^{-12} \end{bmatrix} \quad (4.2)$$

Eigene Prozedur für die Lineare Regression

```
> # Eigene Prozedur für die Lineare Regression
# Geradengleichung: y = a + m*x
LineareRegression:=proc(xListe::list,yListe::list,eyListe::list)
local n, s, sx, sy, sxx, sxy, delta, i,
      a, m, ea, em, rho, chiq,
      p1, p2;
  if (nops(xListe) <> nops(yListe)) then
    printf("nops(xListe)=%d nops(yListe)=%d \n", nops(xListe), nops(yListe));
    error("xListe und yListe müssen die gleiche Anzahl von Elementen haben !");
  end if;
```

Lineare Regression mit Cassy-Lab-Dateien mit Maple

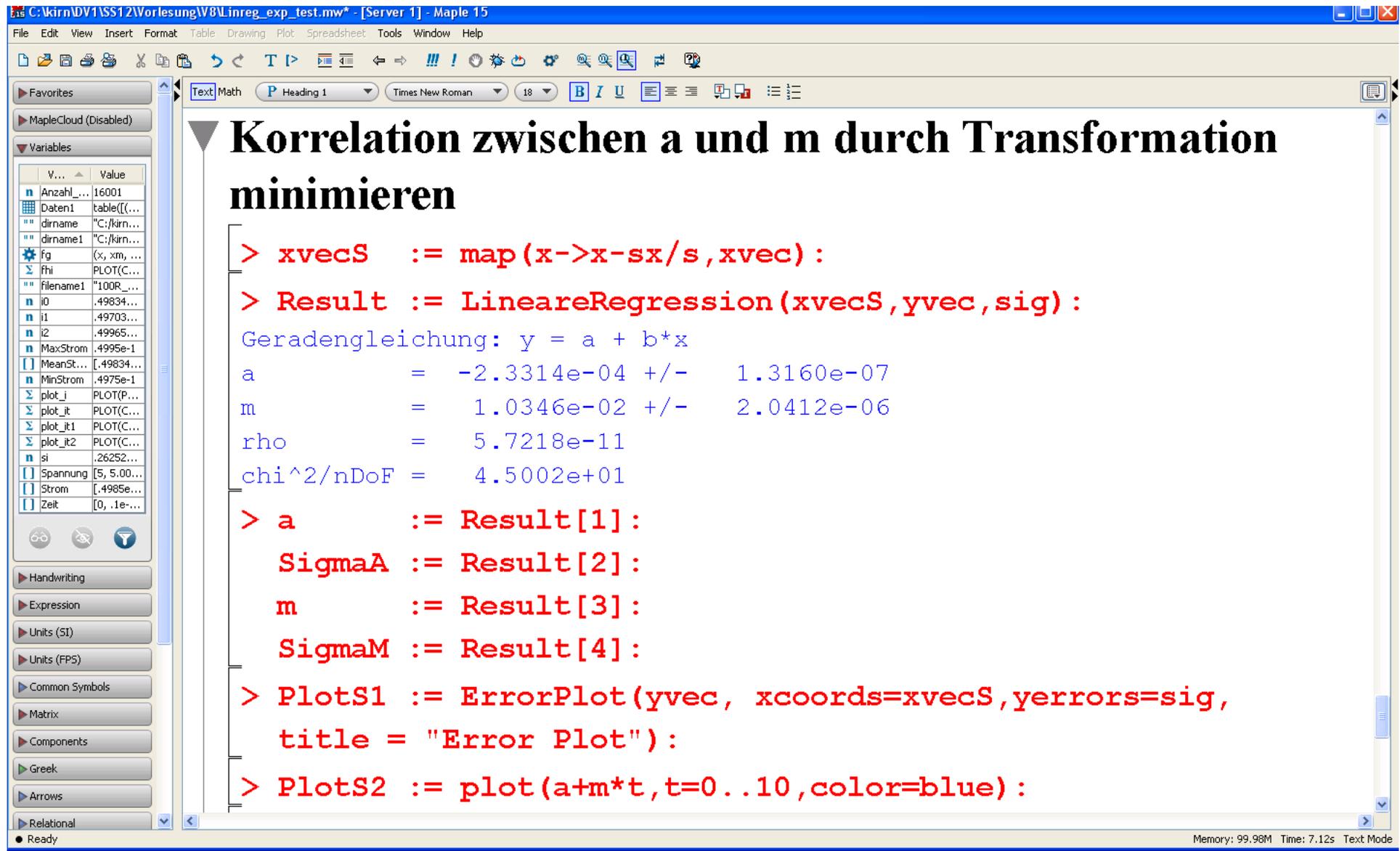


The screenshot shows the Maple 15 software interface. The title bar indicates the file path: C:\Kirn\DV1\SS12\Worlesung\W8\Linreg_exp.mw* - [Server 1] - Maple 15. The interface includes a menu bar (File, Edit, View, Insert, Format, Table, Drawing, Plot, Spreadsheet, Tools, Window, Help), a toolbar with various icons, and a left sidebar with categories like Favorites, Variables, Handwriting, Expression, Units (SI), Units (FPS), Common Symbols, Matrix, Components, Greek, Arrows, Relational, Relational Round, Negated, Large Operators, Operators, Open Face, Fraktur, Script, and Miscellaneous. The main workspace contains a procedure for linear regression, written in red text, and its execution results.

```
n := nops(xListe):
s := sum(1.0/eyListe[i]^2,i=1..n):
sx := sum(xListe[i]/eyListe[i]^2,i=1..n):
sy := sum(yListe[i]/eyListe[i]^2,i=1..n):
sxx := sum(xListe[i]^2/eyListe[i]^2,i=1..n):
sxy := sum(xListe[i]*yListe[i]/eyListe[i]^2,i=1..n):
delta := evalf(s*sxx-sx*sx):
a := evalf((sxx*sy-sx*sxy)/delta):
m := evalf((s*sxy-sx*sy)/delta):
ea := evalf(sqrt(sxx/delta)):
em := evalf(sqrt(s/delta)):
rho := evalf(-sx/sqrt(s*sxx));
chiq := sum((yListe[i]-(a+m*xListe[i]))^2/eyListe[i]^2,i=1..n):
printf("Geradengleichung: y = a + b*x \n"):
printf("a          = %12.4e +/- %12.4e \n",eval(a),eval(ea)):
printf("m          = %12.4e +/- %12.4e \n",eval(m),eval(em)):
printf("rho         = %12.4e \n",eval(rho)):
printf("chi^2/nDoF = %12.4e \n",eval(chiq)/(n-2.0)):
return([a, ea, m, em, chiq, rho]):
end proc:
> Result := LinearRegression(xvec,yvec,sig):
Geradengleichung: y = a + b*x
a          = -4.0512e-04 +/-  1.3591e-07
m          =  1.0346e-02 +/-  2.0412e-06
rho         = -2.4968e-01
chi^2/nDoF =  4.5002e+01
```

Memory: 0.74M Time: 0.10s Text Mode

Lineare Regression mit Cassy-Lab-Dateien mit Maple

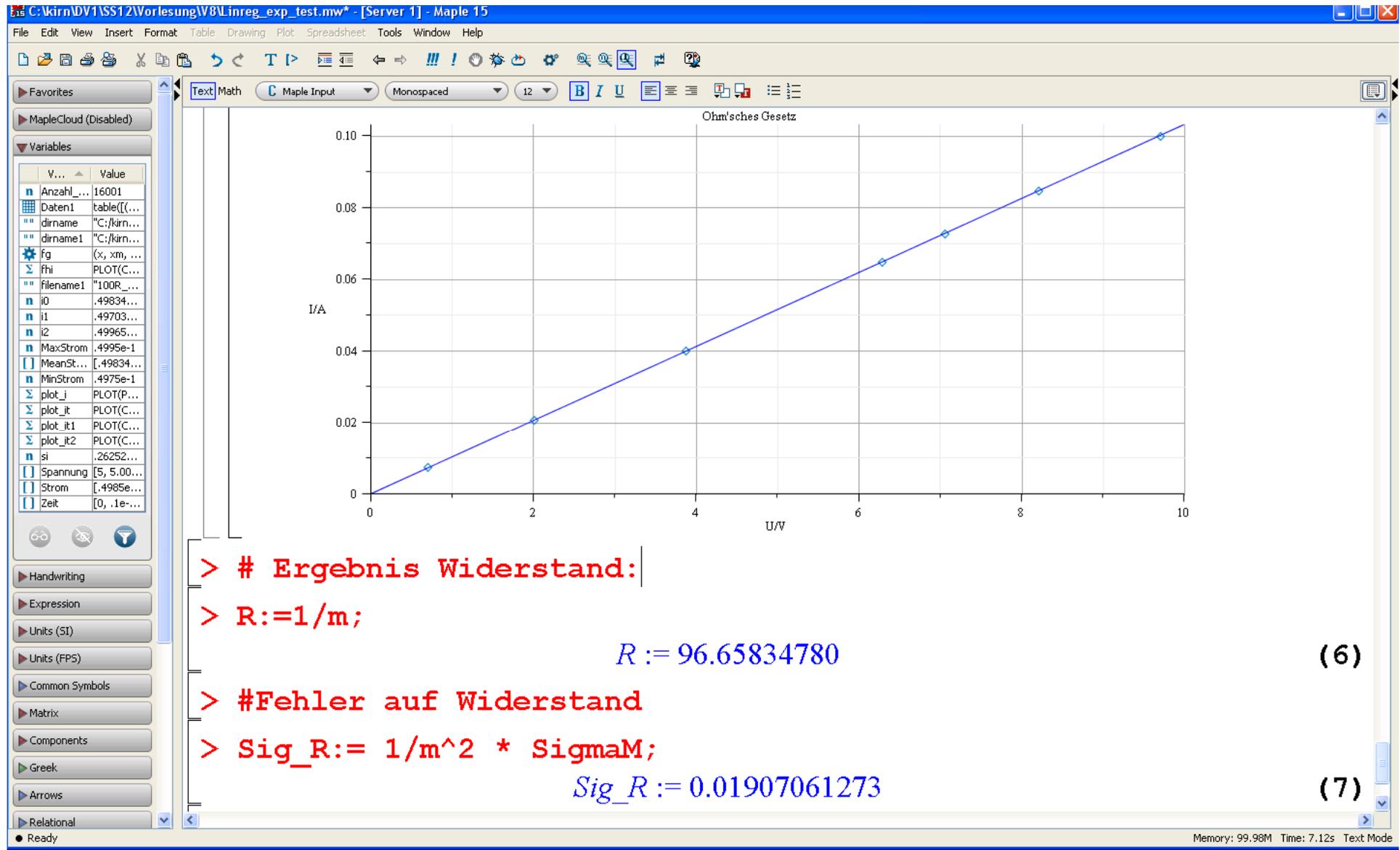


Korrelation zwischen a und m durch Transformation minimieren

```
> xvecS := map(x->x-sx/s, xvec) :
> Result := LinearRegression(xvecS, yvec, sig) :
Geradengleichung: y = a + b*x
a = -2.3314e-04 +/- 1.3160e-07
m = 1.0346e-02 +/- 2.0412e-06
rho = 5.7218e-11
chi^2/nDoF = 4.5002e+01
> a := Result[1]:
SigmaA := Result[2]:
m := Result[3]:
SigmaM := Result[4]:
> PlotS1 := ErrorPlot(yvec, xcoords=xvecS, yerrors=sig,
title = "Error Plot"):
> PlotS2 := plot(a+m*t, t=0..10, color=blue) :
```

| V... | Value |
|------------|-------------|
| Anzahl_... | 16001 |
| Daten1 | table([... |
| dirname | "C:/kirn... |
| dirname1 | "C:/kirn... |
| fg | (x, xm, ... |
| phi | PLOT(C... |
| filename1 | "100R... |
| i0 | .49834... |
| i1 | .49703... |
| i2 | .49965... |
| MaxStrom | .4995e-1 |
| MeanSt... | [.49834... |
| MinStrom | .4975e-1 |
| plot_j | PLOT(P... |
| plot_it | PLOT(C... |
| plot_it1 | PLOT(C... |
| plot_it2 | PLOT(C... |
| si | .26252... |
| Spannung | [5, 5.00... |
| Strom | [.4985e... |
| Zeit | [0, .1e... |

Lineare Regression mit Cassy-Lab-Dateien mit Maple



Anhang 1

$$\sigma_a^2 = \sum_{i=1}^n \sigma_i^2 \left(\frac{\partial a}{\partial y_i} \right)^2 \quad \frac{\partial a}{\partial y_i} = \frac{1}{\Delta \sigma_i^2} (S_{xx} - S_x \cdot x_i)$$

$$\begin{aligned} \Rightarrow \sigma_a^2 &= \sum_{i=1}^n \sigma_i^2 \left(\frac{1}{\Delta \sigma_i^2} (S_{xx} - S_x \cdot x_i) \right)^2 = \sum_{i=1}^n \frac{1}{\Delta^2 \sigma_i^2} (S_{xx}^2 - 2S_{xx} S_x x_i + S_x^2 x_i^2) \\ &= \frac{1}{\Delta^2} (S_{xx}^2 S - 2S_{xx} S_x^2 + S_x^2 S_{xx}) = \frac{1}{\Delta^2} (S_{xx}^2 S - S_{xx} S_x^2) \\ &= \frac{S_{xx}}{\Delta^2} (S \cdot S_{xx} - S_x S_x) = \frac{S_{xx}}{\Delta} \end{aligned}$$

$$S_y \equiv \sum_{i=1}^n \frac{y_i}{\sigma_i^2} \quad S_x \equiv \sum_{i=1}^n \frac{x_i}{\sigma_i^2} \quad S_{xy} \equiv \sum_{i=1}^n \frac{x_i y_i}{\sigma_i^2}$$

$$S_{xx} = \sum_{i=1}^n \frac{x_i^2}{\sigma_i^2} \quad S \equiv \sum_{i=1}^n \frac{1}{\sigma_i^2} \quad \Delta = S \cdot S_{xx} - S_x S_x$$